

ON THE ASYMPTOTIC MEASURE OF PERIODIC SUBSYSTEMS OF FINITE TYPE IN SYMBOLIC DYNAMICS

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ABSTRACT. Let $\Delta \subsetneq V$ be a proper subset of the vertices V of the defining graph of an irreducible and aperiodic shift of finite type (Σ_A^+, T) . Let Δ_n be the union of cylinders in Σ_A^+ corresponding to the points x for which the first n -symbols of x belong to Δ and let μ be an equilibrium state of a Hölder potential φ on Σ_A^+ . We know that $\mu(\Delta_n)$ converges to zero as n diverges. We study the asymptotic behaviour of $\mu(\Delta_n)$ and compare it with the pressure of the restriction of φ to Σ_Δ . The present paper extends some results in [2] to the case when Σ_Δ is irreducible and periodic. We show an explicit example where the asymptotic behaviour differs from the aperiodic case.

INTRODUCTION

The present work was triggered by our study of the convergence of certain hitting time processes to (marked) Poisson processes [2]. The general setup in the study of asymptotic time distributions for an ergodic dynamical system (Ω, μ, T) is the following (see e.g. [1]). Usually one considers a sequence of measurable subsets B_n with asymptotically vanishing measure and one tries to study the asymptotic behaviour of the random variables:

$$\tau_n(\omega) = \inf\{k \geq 1 : T^k(\omega) \in B_n\}$$

when suitably rescaled. The rescaling is necessary since in general $\tau_n(\omega) \rightarrow \infty$ for μ -almost every ω . The most studied case is when Ω is equipped with a partition and $B_n = B_n(\omega)$ are cylinder sets about a generic point ω of μ (so that $\cap_n B_n = \omega$). It turns out that $\mu(B_n)$ is the right scaling factor to obtain, for “sufficiently” mixing systems, a convergence to a Poisson law for the point process associated with the random variables $\mu(B_n)\tau_n$. The reader can find a list of references in e.g. [2]. We were interested in the case where the intersection of the B_n ’s is a non-trivial invariant set. In [2] we consider the case where

Date: April 16, 2008.

2000 Mathematics Subject Classification. Primary 37D35; Secondary 47A35.

Key words and phrases. pressure, Gibbs states, Pianigiani-Yorke measure.

that intersection is a subsystem of finite type of a shift of finite type. This is closely related with the so-called Pianigiani-Yorke measures [3].

More precisely, the setup is as follows. Let $\Delta \subsetneq \mathbf{V}$ be a proper subset of the vertices \mathbf{V} of the defining graph of an irreducible and aperiodic shift of finite type (Σ_A^+, T) . This induces a subsystem of finite type which we denote by Σ_Δ . Let Δ_n be the union of cylinders in Σ_A^+ corresponding to the points x for which the first n -symbols of x belong to Δ . Note that $\Sigma_\Delta = \bigcap_{n \geq 0} \Delta_n$. Now let μ be an equilibrium state of a Hölder potential φ on Σ_A^+ . We know that $\mu(\Delta_n)$ converges to zero as n diverges. In [2] we assume that Σ_Δ is an irreducible and *aperiodic* subshift of finite type. Then we prove that the point process corresponding to the times of hitting Δ_n scaled by $\mu(\Delta_n)$ converges to a (marked) Poisson point process. In that paper, we study the asymptotic behaviour of $\mu(\Delta_n)$ and compare it with the pressure of the restriction of φ to Σ_Δ , call it P_Δ . We show that $e^{n(P(\varphi)-P_\Delta)}\mu(\Delta_n)$ has a limit (which can be identified) as $n \rightarrow \infty$. In the present paper we extend some results in [2] to the case when Σ_Δ is irreducible *but periodic*. This extension turns out to be non-trivial since, in general, $e^{n(P(\varphi)-P_\Delta)}\mu(\Delta_n)$ may *not converge* as $n \rightarrow \infty$, contrarily to the aperiodic case. Indeed, we provide an explicit example where this phenomenon appears.

1. PRELIMINARIES

Let $\mathbf{V} = \{1, \dots, \ell\}$ be a finite set of symbols (i.e. the base alphabet). We will assume that A is an aperiodic 0-1 $\ell \times \ell$ matrix which defines the allowable transitions in a directed graph \mathcal{G} of labelled vertices \mathbf{V} . Define the space of one-sided allowable paths in the graph \mathcal{G} by

$$\Sigma_A^+ = \{x = (x_n) \in \mathbf{V}^{\mathbb{N}} : A(x_{i-1}, x_i) = 1, \forall i \geq 1\}.$$

The space Σ_A^+ is compact and metrisable when endowed with the Tychonov product topology (generated by the discrete topology on \mathbf{V}). The *shift* T (of finite type) is the map $T: \Sigma_A^+ \rightarrow \Sigma_A^+$ defined by $T(x)_n = x_{n+1}$ for all $n \geq 0$. This map is continuous and surjective. The cylinders, denoted by

$$C[i_0, \dots, i_m]_k = \{x \in \Sigma_A^+ : x_{j+k} = i_j, \forall j = 0, \dots, m\},$$

form a base of open (and closed) sets in Σ_A^+ . Let $C(\Sigma_A^+)$ denote the space of complex valued continuous functions on Σ_A^+ . For $\psi \in C(\Sigma_A^+)$, consider $\text{var}_n(\psi) = \sup\{|\psi(x) - \psi(y)| : x_i = y_i, i \leq n\}$. Given $0 < \theta < 1$, define $|\psi|_\theta = \sup\{\text{var}_n(\psi)/\theta^n\}$. The space $\mathcal{F}_\theta^+ = \{\psi \in C(\Sigma_A^+) : |\psi|_\theta < \infty\}$ is a Banach space when endowed with the norm $\|\psi\|_\theta = \|\psi\|_\infty + |\psi|_\theta$, where $\|\cdot\|_\infty$ denotes the supremum norm. The

union $\mathcal{F} = \cup_{\theta} \mathcal{F}_{\theta}$ is referred to as the space of Hölder continuous functions on Σ_A^+ .

Let $\varphi \in \mathcal{F}_{\theta}^+$ be a real valued function. Define the transfer operator L_{φ} acting on \mathcal{F}_{θ}^+ by

$$(L_{\varphi}\psi)(x) = \sum_{Ty=x} e^{\varphi(y)} \psi(y) .$$

The operator L_{φ} has a maximum positive eigenvalue $e^{P(\varphi)}$, which is simple and isolated. Furthermore, the rest of the spectrum is contained in a disc of radius strictly less than $e^{P(\varphi)}$ (cf. [5], [4]). The number $P = P(\varphi)$ is called the *pressure* of φ . There is a unique T -invariant probability measure $\mu = \mu_{\varphi}$ such that

$$P(\varphi) = h(\mu) + \int \varphi d\mu ,$$

where $h(\mu)$ denotes the measure-theoretic entropy of (T, μ) . The pressure $P(\varphi)$ can also be characterised as the maximum of $h(m) + \int \varphi dm$ over all T -invariant probabilities m . The measure μ is called the *equilibrium state* of φ . An eigenfunction w of L_{φ} corresponding to $e^{P(\varphi)}$ may be taken to be strictly positive, in fact one may take w to be the function

$$w = \lim_{n \rightarrow \infty} e^{-nP(\varphi)} L_{\varphi}^n(\mathbf{1}) ,$$

where $\mathbf{1}$ denotes the constant function equal to 1. Replacing φ by $\varphi' = \varphi - P(\varphi) + \log(w) - \log(w \circ T)$, we see that $L_{\varphi'} \mathbf{1} = \mathbf{1}$ and $P(\varphi') = 0$. In this case we say that φ' is *normalised*. It is easy to see that φ and φ' have the same equilibrium state μ . In what follows we will assume that φ is normalised. Note that in this case the transfer operator L_{φ} satisfies

$$\begin{aligned} \int \psi d\mu &= \int L_{\varphi}(\psi) d\mu , \\ \int \psi_1 \cdot (\psi_2 \circ T) d\mu &= \int L_{\varphi}(\psi_1) \cdot \psi_2 d\mu , \end{aligned}$$

for all $\psi, \psi_1, \psi_2 \in C(\Sigma_A^+)$.

Let $\Delta \subseteq \mathbf{V}$ be a sub-alphabet such that $\Delta \neq \mathbf{V}$. Consider the closed T -invariant subset $\Sigma_{\Delta} \subseteq \Sigma_A^+$ given by

$$\Sigma_{\Delta} = \{x \in \Sigma_A^+ : x_i \in \Delta, \forall i \geq 0\} .$$

In this paper we will consider only the case when Σ_{Δ} is an irreducible subshift of finite type in its alphabet Δ . This means that the restriction of the matrix A to the symbols of Δ defines a matrix A_{Δ} which is irreducible. In particular, the restriction of the shift transformation T to Σ_{Δ} is topologically transitive in the induced topology from Σ_A^+ .

Let φ_Δ denote the restriction of φ to the subsystem Σ_Δ . Let P_Δ be the pressure of φ_Δ with respect to the subsystem (Σ_Δ, T) . (Note that since φ is assumed to be normalised we have $P(\varphi) = 0$, therefore $P_\Delta < 0$.) Let μ_Δ denote the equilibrium state of φ_Δ with respect to the subsystem (Σ_Δ, T) . Let w_Δ be the strictly positive Hölder continuous function defined on Σ_Δ by

$$(1) \quad w_\Delta = \lim_{n \rightarrow \infty} e^{-nP_\Delta} L_{\varphi_\Delta}^n(\mathbf{1}) .$$

Now define the *restricted transfer operator* L_Δ acting on the space of Hölder continuous functions \mathcal{F}_θ^+ by

$$L_\Delta \psi = L_\varphi(\psi \cdot \chi_\Delta) ,$$

and consider the subset of Σ_A^+ given by

$$(2) \quad \mathcal{Z}_\Delta = \{x \in \Sigma_A^+ : \exists b \in \Delta, A(b, x_0) = 1\} .$$

Note that since A is irreducible and aperiodic in the full alphabet \mathbf{V} , \mathcal{Z}_Δ is a non-empty finite union of cylinder sets of Σ_Δ^+ . In particular, since μ is fully supported on Σ_A^+ we have $\mu(\mathcal{Z}_\Delta) > 0$.

An improvement to main result of [3], which is proved in [2], gives the following result.

Proposition 1. *There exists a unique Hölder continuous function h_Δ defined on the whole space Σ_A^+ such that*

$$L_\Delta(h_\Delta) = e^{P_\Delta} h_\Delta ,$$

and $h_\Delta|_{\Sigma_\Delta} \equiv w_\Delta$, where w_Δ is given by (1). The function h_Δ is strictly positive on \mathcal{Z}_Δ and it is zero on the complement \mathcal{Z}_Δ^c . Moreover,

$$\left\| e^{-nP_\Delta} L_\Delta^n(\psi) - h_\Delta \int_{\Sigma_\Delta} \psi d\mu_\Delta \right\|_{\Sigma_A^+} \xrightarrow{n \rightarrow \infty} 0 ,$$

for all $\psi \in C(\Sigma_A^+)$.

The Borel measure μ_{BPY} defined by

$$\mu_{\text{BPY}}(B) = \int_B h_\Delta d\mu$$

for every Borel set $B \subseteq \Sigma_A^+$ is called the *Pianigiani-Yorke measure* of the subsystem (Σ_Δ, T) . This measure is fully supported on \mathcal{Z}_Δ .

The following is a result from [2].

Proposition 2. *Let h_Δ be the function in Proposition 1. We have*

$$\lim_{n \rightarrow \infty} e^{-nP_\Delta} \mu(\Delta_n) = \int h_\Delta d\mu = \mu_{\text{BPY}}(\Sigma_A^+) .$$

2. THE PERIODIC CASE

Let us consider the case when Σ_Δ is irreducible but periodic with period $m > 1$. In this case there exists a decomposition of $\Delta = \Delta_0 \cup \dots \cup \Delta_{m-1}$ with the property that if $i \in \Delta_s, j \in \Delta_{s'}$ are given such that $A(i, j) = 1$ then necessarily $s' = s+1 \pmod{m}$. This induces a disjoint partition of $\Sigma_\Delta = \Omega_0 \cup \dots \cup \Omega_{m-1}$ such that $T(\Omega_s) = \Omega_{s+1 \pmod{m}}$, which is the so-called cyclically moving partition of Σ_Δ . From the classical Ruelle-Perron-Frobenius theory (cf. [5], [4]), we know that there exist non-negative Hölder continuous functions w_0, \dots, w_{m-1} defined on Σ_Δ , and mutually singular probability measures¹ ν_0, \dots, ν_{m-1} with ν_j supported on Ω_j satisfying

$$\mathbb{L}_{\varphi_\Delta}(w_j) = e^{P_\Delta} w_{j+1 \pmod{m}} ,$$

and $\text{supp}(w_j) = \Omega_j$ for $j = 0, \dots, m-1$. Moreover, $w_\Delta = \sum_{j=0}^{m-1} w_j$ is strictly positive on Σ_Δ and

$$\left\| e^{-nP_\Delta} \mathbb{L}_{\varphi_\Delta}^n(\psi) - \sum_{j=0}^{m-1} w_{j+n \pmod{m}} \int_{\Omega_j} \psi d\nu_j \right\|_{\Sigma_\Delta} \xrightarrow{n \rightarrow \infty} 0 ,$$

for all $\psi \in C(\Sigma_\Delta)$. Putting $\psi = \mathbf{1}$ and replacing n by nm in the above expression, we note that w_Δ could have been defined uniquely by

$$w_\Delta = \lim_{n \rightarrow \infty} e^{-nmP_\Delta} \mathbb{L}_{\varphi_\Delta}^{nm}(\mathbf{1}) ,$$

and then $\mathbb{L}_{\varphi_\Delta}(w_\Delta) = e^{P_\Delta} w_\Delta$. Now we transfer these results from Σ_Δ to the whole space Σ_A^+ . Let \mathcal{Z}_Δ be defined by (2). Since again \mathcal{Z}_Δ is a non-empty finite union of cylinders of Σ_A^+ , we have $\mu(\mathcal{Z}_\Delta) > 0$.

Define the constants d_j by

$$d_j = \int_{\Omega_{j+1 \pmod{m}}} \mathbb{L}_\Delta(\mathbf{1}) d\nu_{j+1 \pmod{m}} ,$$

for $j = 0, \dots, m-1$, we see that $d_j > 0$ for all j . Define also the constants $\alpha_j(k)$ by $\alpha_j(0) = 1$ and for $1 \leq k \leq m-1$,

$$\alpha_j(k) = e^{-kP_\Delta} \prod_{s=0}^{k-1} d_{j+s \pmod{m}} ,$$

for $j = 0, \dots, m-1$. The next is our main result.

Theorem 3. *There exist a unique choice of non-negative Hölder continuous functions h_0, \dots, h_{m-1} defined on the whole space Σ_A^+ satisfying*

$$L_\Delta(h_j) = e^{P_\Delta} h_{j+1 \pmod{m}} ,$$

¹In fact, ν_j is the equilibrium state of the potential $S_m(\varphi) = \sum_{i=0}^{m-1} \varphi \circ T^i$ restricted to (Ω_j, T^m) and $\nu_{j+1 \pmod{m}} = \nu_j \circ T^{-1}$.

and $h_j|_{\Omega_j} \equiv w_j$ for $j = 0, \dots, m-1$. The function $h_\Delta = \sum_{j=0}^{m-1} h_j$ is strictly positive on \mathcal{Z}_Δ and it is zero on the complement \mathcal{Z}_Δ^c . Moreover,

$$(3) \quad \left\| e^{-nP_\Delta} L_\Delta^n(\psi) - \sum_{j=0}^{m-1} \alpha_j(n \pmod{m}) h_{j+n \pmod{m}} \int_{\Omega_j} \psi d\nu_j \right\|_{\Sigma_A^+} \xrightarrow{n \rightarrow \infty} 0,$$

for all $\psi \in \mathcal{C}(\Sigma_A^+)$.

In particular taking $\psi = 1$ and integrating the above expression with respect to μ we obtain

Corollary 4. *Let Σ_Δ be an irreducible and periodic subsystem of finite type with period m . The sets Δ_n have the following asymptotic behaviour:*

$$\left| e^{-nP_\Delta} \mu(\Delta_n) - \sum_{j=0}^{m-1} \alpha_j(n \pmod{m}) \int h_{j+n \pmod{m}} d\mu \right| \xrightarrow{n \rightarrow \infty} 0.$$

In particular, for each $k = 0, 1, \dots, m-1$ we have

$$\lim_{n \rightarrow \infty} e^{-(k+nm)P_\Delta} \mu(\Delta_{k+nm}) = \sum_{j=0}^{m-1} \alpha_j(k) \int h_{j+k \pmod{m}} d\mu.$$

In Section 4 we give an explicit example where the above numbers differ for different choices of k , which shows that $e^{-nP_\Delta} \mu(\Delta_n)$ does not converge in general as $n \rightarrow \infty$ when Σ_Δ is periodic.

3. PROOF OF THEOREM 3

We recall some results from [3]. Let $\mathcal{C}_p^+(\Sigma_A^+)$ be the set of strictly positive p -cylindrical functions (i.e. a function depending only on the first p coordinates of the point). Let $0 < \theta < 1$ be the Hölder exponent of the potential φ . Let \mathcal{Z}_Δ be defined as in (2). Let $\mathcal{C}(\mathcal{Z}_\Delta)$ denote the set of continuous functions defined on \mathcal{Z}_Δ . The proof of the following Lemma can be obtained from Appendix C in [2].

Lemma 5. *For any $f \in \cup_{p \geq 1} \mathcal{C}_p^+(\Sigma_A^+)$, we have*

- (i) $\{e^{-nP_\Delta} L_\Delta^n f\}_{n \geq 0}$ is a Cauchy sequence in $\mathcal{C}(\Sigma_A^+)$;
- (ii) $h_\Delta = \lim_{n \rightarrow \infty} \frac{e^{-nP_\Delta} L_\Delta^n f}{\int f d\mu_\Delta}$ does not depend on the function $f \in \cup_{p \geq 1} \mathcal{C}_p^+(\Sigma_A^+)$ and it satisfies

$$L_\Delta(h_\Delta) = e^{P_\Delta} h_\Delta.$$

Although not explicitly mentioned in [3], the function h_Δ is a Hölder continuous function with the same Hölder exponent of the potential φ . We also note that, since μ is fixed by the dual operator of L we have

$$(4) \quad \int_{\Delta_n} f \cdot g \circ T^n d\mu = \int L_\varphi^n(\chi_{\Delta_n} \cdot f \cdot g \circ T^n) d\mu = \int g \cdot L_\Delta^n(f) d\mu.$$

Now consider the case when Σ_Δ is irreducible but periodic with period $m > 1$. Consider the decomposition of $\Delta = \Delta_0 \cup \dots \cup \Delta_{m-1}$ with the property that if $i \in \Delta_s, j \in \Delta_{s'}$ are given such that $A(i, j) = 1$ then necessarily $s' = s + 1 \pmod{m}$. Consider also the corresponding cyclically moving partition $\Sigma_\Delta = \Omega_0 \cup \dots \cup \Omega_{m-1}$ such that $T(\Omega_s) = \Omega_{s+1 \pmod{m}}$, i.e. defining

$$\Omega_j = \{x \in \Sigma_\Delta : x_0 \in \Delta_j\},$$

for $j = 0, \dots, m-1$. Let $\mathbf{V}^{(m)}$ be the sub-alphabet of \mathbf{V}^m defined by

$$\mathbf{V}^{(m)} = \{(i_0, \dots, i_{m-1}) \in \mathbf{V}^m : i_0 \rightarrow \dots \rightarrow i_{m-1} \text{ in } \mathcal{G}\},$$

where \mathcal{G} is the defining graph of Σ_A^+ . Consider the transition matrix $A^{(m)}$ indexed by $\mathbf{V}^{(m)} \times \mathbf{V}^{(m)}$ given by

$$A^{(m)}((i_0, \dots, i_{m-1}), (j_0, \dots, j_{m-1})) = 1 \quad \text{if } i_{m-1} \rightarrow j_0 \text{ in } \mathcal{G}.$$

Using the identification

$$(5) \quad ((x_0, \dots, x_{m-1}), (x_m, \dots, x_{2m-1}), \dots) \longleftrightarrow (x_0, x_1, x_2, \dots),$$

the shift transformation T_m on $\Sigma_{A^{(m)}}^+$ is naturally topologically conjugate to T^m on Σ_A^+ . In what follows we will abuse the notation and freely identify these transformations and spaces.

The normalised potential φ on Σ_A^+ naturally defines a potential $\varphi^{(m)}$ on $\Sigma_{A^{(m)}}^+$ by $\varphi^{(m)} = S_m(\varphi) = \sum_{j=0}^{m-1} \varphi \circ T^j$. Note that $\varphi^{(m)}$ is a normalised potential for T_m , i.e.

$$L_{\varphi^{(m)}}(\mathbf{1})(x) = \sum_{y \in T_m^{-1}(x)} e^{\varphi^{(m)}(y)} = 1,$$

for all $x \in \Sigma_{A^{(m)}}^+$. Now the sub-alphabet Δ of \mathbf{V} defines a sub-alphabet $\Delta^{(m)}$ of $\mathbf{V}^{(m)}$ by

$$\Delta^{(m)} = \{(i_0, \dots, i_{m-1}) \in \mathbf{V}^{(m)} : i_s \in \Delta, \text{ for } s = 0, \dots, m-1\},$$

and this sub-alphabet can be further decomposed into

$$(6) \quad \Delta_j^{(m)} = \{(i_0, \dots, i_{m-1}) \in \Delta^{(m)} : i_s \in \Delta_{s+j \pmod{m}}, \text{ for } s = 0, \dots, m-1\},$$

for $j = 0, \dots, m-1$. The important fact is that for fixed j , $\Sigma_j^{(m)}$ the subsystem of $\Sigma_{A^{(m)}}^+$ obtained by taking transitions through $\Delta_j^{(m)}$ is

irreducible and aperiodic in its alphabet $\Delta_j^{(m)}$. Hence the main result of [3] applies and we define a Hölder continuous function h_j by

$$(7) \quad h_j = \lim_{n \rightarrow \infty} e^{-nP(\Delta_j^{(m)})} \mathbb{L}_{\Delta_j^{(m)}}^n(\mathbf{1}) ,$$

where $P(\Delta_j^{(m)})$ is the pressure of the restriction of $\varphi^{(m)}$ to the subsystem $\Sigma_j^{(m)}$ (hence $P(\Delta_j^{(m)}) = mP_\Delta$ for all j), and $\mathbb{L}_{\Delta_j^{(m)}}(\psi) = \mathbb{L}_{\varphi^{(m)}}(\psi \cdot \chi_{\Delta_j^{(m)}})$ with respect to the shift T_m . From Lemma 5 (ii) extended to continuous functions we obtain

$$(8) \quad \lim_{n \rightarrow \infty} e^{-nmP_\Delta} \mathbb{L}_{\Delta_j^{(m)}}^n(\psi) = h_j \int_{\Sigma_j^{(m)}} \psi d\nu_j ,$$

for every $\psi \in \mathcal{C}(\Sigma_{A^{(m)}}^+)$ (with the limit being uniform), where ν_j is the unique equilibrium state of $\varphi^{(m)}$ restricted to the subsystem $\Sigma_j^{(m)}$ with respect to the shift T_m .

In view of the identification (5) the function h_j defines a function on Σ_A^+ in a natural way and ν_j becomes a probability measure on Σ_A^+ fully supported on Ω_j . Note that h_j is then strictly positive on

$$\mathcal{Z}_{\Delta_j} = \{x \in \Sigma_A^+ : \exists b \in \Delta_{j-1 \pmod m}, A(b, x_0) = 1\} ,$$

and it is zero on the complement $\mathcal{Z}_{\Delta_j}^c$. Note also that for each j , \mathcal{Z}_{Δ_j} is a non-empty finite union of cylinders of Σ_A^+ , therefore in particular, $\mu(\mathcal{Z}_{\Delta_j}) > 0$. Applying \mathbb{L}_Δ as

$$\begin{aligned} \mathbb{L}_\Delta(\psi)((x_0, \dots, x_{m-1}), (x_m, \dots, x_{2m-1}), \dots) \\ = \sum_{\{i \in \Delta : A(i, x_0)=1\}} e^{\varphi^{(m)}((i, x_0, \dots, x_{m-2}), (x_{m-1}, \dots, x_{2m-2}), \dots)} \times \\ \psi((i, x_0, \dots, x_{m-2}), (x_{m-1}, \dots, x_{2m-2}), \dots) , \end{aligned}$$

we conclude that $\mathbb{L}_\Delta \circ \mathbb{L}_{\Delta_j^{(m)}}^n = \mathbb{L}_{\Delta_{j+1 \pmod m}^{(m)}}^n \circ \mathbb{L}_\Delta$. This implies that for all $k \geq 1$ we have $\mathbb{L}_\Delta^k \circ \mathbb{L}_{\Delta_j^{(m)}}^n = \mathbb{L}_{\Delta_{j+k \pmod m}^{(m)}}^n \circ \mathbb{L}_\Delta^k$. Putting $\psi = \mathbf{1}$ in (8) we obtain, for fixed $k \geq 1$ and fixed $0 \leq j < m$,

$$\begin{aligned} (9) \quad \mathbb{L}_\Delta^k(h_j) &= \lim_{n \rightarrow \infty} e^{-nmP_\Delta} \mathbb{L}_\Delta^k(\mathbb{L}_{\Delta_j^{(m)}}^n(\mathbf{1})) = \lim_{n \rightarrow \infty} e^{-nmP_\Delta} \mathbb{L}_{\Delta_{j+k \pmod m}^{(m)}}^n(\mathbb{L}_\Delta^k(\mathbf{1})) \\ &= h_{j+k \pmod m} \int_{\Sigma_{j+k \pmod m}^{(m)}} \mathbb{L}_\Delta^k(\mathbf{1}) d\nu_{j+k \pmod m} . \end{aligned}$$

Therefore defining the constants d_j by

$$d_j = \int_{\Omega_{j+1 \pmod m}} \mathbb{L}_\Delta(\mathbf{1}) d\nu_{j+1 \pmod m} ,$$

for $j = 0, \dots, m-1$, we see that $d_j > 0$ for all j and by (9) we have

$$\mathbf{L}_\Delta(h_j) = d_j h_{j+1 \pmod{m}} ,$$

for all j . Since $\mathbf{L}_\Delta^m(h_j) = e^{mP_\Delta} h_j$ for each j , by (9) we also see that

$$\prod_{j=0}^{m-1} d_j = e^{mP_\Delta} .$$

At the end of this appendix we give an example where in general one has d_j not necessarily equal to e^{P_Δ} .

The function $h_\Delta = \sum_{j=0}^{m-1} h_j$ is strictly positive on \mathcal{Z}_Δ and it is zero on the complement \mathcal{Z}_Δ^c . (Note that $\mathcal{Z}_\Delta = \cup_{j=0}^{m-1} \mathcal{Z}_{\Delta_j}$ and this union is not in general a disjoint union, see example below.) The function h_Δ also satisfies

$$\mathbf{L}_\Delta^m(h_\Delta) = e^{mP_\Delta} h_\Delta .$$

Now, from the fact that

$$\mathbf{L}_{\Delta^{(m)}}(\psi) = \sum_{j=0}^{m-1} \mathbf{L}_{\Delta_j^{(m)}}(\psi) ,$$

and $\mathbf{L}_{\Delta_j^{(m)}} \circ \mathbf{L}_{\Delta_{j'}^{(m)}} = 0$ if $j \neq j'$, we see that

$$\mathbf{L}_{\Delta^{(m)}}^n(\psi) = \sum_{j=0}^{m-1} \mathbf{L}_{\Delta_j^{(m)}}^n(\psi) .$$

Using (9) we have, for fixed $1 \leq k < m$,

$$\begin{aligned} (10) \quad e^{-(nm+k)P_\Delta} \mathbf{L}_\Delta^{nm+k}(\psi) &= e^{-(nm+k)P_\Delta} \mathbf{L}_\Delta^k(\mathbf{L}_{\Delta^{(m)}}^n(\psi)) \\ &= \sum_{j=0}^{m-1} e^{-kP_\Delta} \mathbf{L}_\Delta^k(e^{-nmP_\Delta} \mathbf{L}_{\Delta_j^{(m)}}^n(\psi)) \\ &= \sum_{j=0}^{m-1} e^{-kP_\Delta} \mathbf{L}_\Delta^k(h_j) \int_{\Omega_j} \psi d\nu_j + o(1) \\ &= \sum_{j=0}^{m-1} \left(e^{-kP_\Delta} \prod_{s=0}^{k-1} d_{j+s \pmod{m}} \right) h_{j+k \pmod{m}} \int_{\Omega_j} \psi d\nu_j + o(1) , \end{aligned}$$

where we have used Lemma 5 (ii) for continuous functions and $o(1)$ is with respect to n . Define the constants $\alpha_j(k)$ by $\alpha_j(0) = 1$ and for $1 \leq k \leq m-1$,

$$\alpha_j(k) = e^{-kP_\Delta} \prod_{s=0}^{k-1} d_{j+s \pmod{m}} ,$$

for $j = 0, \dots, m-1$. Therefore from (10) we finally obtain

$$\left\| e^{-nP_\Delta} \mathbf{L}_\Delta^n(\psi) - \sum_{j=0}^{m-1} \alpha_j(n \pmod{m}) h_{j+n \pmod{m}} \int_{\Omega_j} \psi d\nu_j \right\|_{\Sigma_A^+} \xrightarrow{n \rightarrow \infty} 0 ,$$

for all $\psi \in \mathcal{C}(\Sigma_A^+)$, which concludes the proof of Theorem 3.

4. ILLUSTRATIVE EXAMPLE

In this section we give an example to illustrate the computations made in the previous section.

Example. Let $\mathbf{V} = \{1, 2, 3\}$ and consider the matrix A given by

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} .$$

Let φ be any normalised Hölder continuous potential on Σ_A^+ , i.e. assume that

$$\mathbf{L}_\varphi(\mathbf{1})(x) = \sum_{\{i \in \mathbf{V}: A(i, x_0)=1\}} e^{\varphi(ix)} = 1 ,$$

for all $x \in \Sigma_A^+$. Take $\Delta = \{1, 2\}$. Then Σ_Δ is the periodic orbit $\{(1, 2, 1, 2, \dots), (2, 1, 2, 1, \dots)\}$. Put $\varphi_\Delta(1, 2, 1, 2, \dots) = p$ and $\varphi_\Delta(2, 1, 2, 1, \dots) = q$, and assume $p \neq q$. There is only one invariant measure for the restriction of the shift T on Σ_Δ , namely

$$\mu_\Delta = \frac{1}{2}(\delta_1 + \delta_2) ,$$

where δ_1 is Dirac measure at the point $(1, 2, 1, 2, \dots)$ and δ_2 is Dirac measure at the point $(2, 1, 2, 1, \dots)$. Since the shift entropy of μ_Δ is zero, the restricted pressure P_Δ is then given by

$$P_\Delta = \int \varphi_\Delta d\mu_\Delta = \frac{1}{2}(p + q) .$$

Notice now that $\varphi^{(2)} = \varphi + \varphi \circ T$ when restricted to Σ_Δ is constant with value $p + q$. The pressure of the restriction of $\varphi^{(2)}$ to Σ_Δ with respect to T^2 is then given by $p + q = 2P_\Delta$. The set Δ is further decomposed into $\Delta_{i-1} = \{i\}$, for $i = 1, 2$, giving the cyclically moving partition $\Sigma_\Delta = \Omega_0 \cup \Omega_1$, where $\Omega_0 = \{(1, 2, 1, 2, \dots)\}$ and $\Omega_1 = \{(2, 1, 2, 1, \dots)\}$. Note then that T^2 restricted to Σ_Δ consists of two fixed points. This implies that $\nu_0 = \delta_1$ and $\nu_1 = \delta_2$, where ν_i is the equilibrium state of $\varphi^{(2)}$ restricted to Ω_i with respect to T^2 . Applying [3] in the case of an

aperiodic subsystem consisting of a fixed point for T^2 we have from (7), where $\Delta_j^{(m)}$ is defined by (6) and $m = 2$,

$$h_j = \lim_{n \rightarrow \infty} e^{-2nP\Delta} \mathbb{L}_{\Delta_j^{(2)}}^n(\mathbf{1}) ,$$

for $j = 0, 1$. Interpreting this we conclude that for $x \in \mathcal{Z}_{\Delta_0} = C[1]_0 \cup C[3]_0$ we have

$$\begin{aligned} h_0(x) &= \lim_{n \rightarrow \infty} \exp \left\{ S_{2n}(\varphi) \left(\underbrace{1, 2, 1, 2, \dots, 1, 2}_{2n}, x_0, x_1, \dots \right) - n(p+q) \right\} \\ &= \lim_{n \rightarrow \infty} \exp \left\{ S_{2n}(\varphi) \left(\underbrace{1, 2, 1, 2, \dots, 1, 2}_{2n}, x_0, x_1, \dots \right) - S_{2n}(\varphi)(1, 2, 1, 2, \dots) \right\} , \end{aligned}$$

where $S_k(\varphi)$ denotes $\varphi + \varphi \circ T + \dots + \varphi \circ T^{k-1}$, and h_0 is zero on the complement $\mathcal{Z}_{\Delta_0}^c = C[2]_0$. Also if $x \in \mathcal{Z}_{\Delta_1} = C[2]_0 \cup C[3]_0$ then

$$\begin{aligned} h_1(x) &= \lim_{n \rightarrow \infty} \exp \left\{ S_{2n}(\varphi) \left(\underbrace{2, 1, 2, 1, \dots, 2, 1}_{2n}, x_0, x_1, \dots \right) - n(p+q) \right\} \\ &= \lim_{n \rightarrow \infty} \exp \left\{ S_{2n}(\varphi) \left(\underbrace{2, 1, 2, 1, \dots, 2, 1}_{2n}, x_0, x_1, \dots \right) - S_{2n}(\varphi)(2, 1, 2, 1, \dots) \right\} , \end{aligned}$$

and h_1 is zero on the complement $\mathcal{Z}_{\Delta_1}^c = C[1]_0$. (Note that h_0 and h_1 are both strictly positive on the cylinder $C[3]_0$.) Now we compute the constants d_j , for $j = 0, 1$. We have

$$\begin{aligned} d_0 &= \int_{\Omega_1} \mathbb{L}_{\Delta}(\mathbf{1}) d\nu_1 = e^{\varphi(1,2,1,2,\dots)} = e^p , \quad \text{and} \\ d_1 &= \int_{\Omega_0} \mathbb{L}_{\Delta}(\mathbf{1}) d\nu_0 = e^{\varphi(2,1,2,1,\dots)} = e^q . \end{aligned}$$

This provides an example where $d_j \neq e^{P\Delta} = e^{\frac{1}{2}(p+q)}$, since we are assuming $p \neq q$. One can see directly that h_0 and h_1 satisfy

$$\mathbb{L}_{\Delta}(h_0) = d_0 h_1 = e^p h_1 \quad \text{and} \quad \mathbb{L}_{\Delta}(h_1) = d_1 h_0 = e^q h_0 .$$

The function $h_{\Delta} = h_0 + h_1$ satisfies $\mathbb{L}_{\Delta}^2(h_{\Delta}) = e^{2P\Delta} h_{\Delta}$, and in the case of this example it is fully supported on Σ_A^+ . Now we compute the constants $\alpha_j(k)$ for $j, k = 0, 1$. We have $\alpha_j(0) = 1$ for $j = 0, 1$,

$$\begin{aligned} \alpha_0(1) &= e^{-P\Delta} d_0 = e^{-\frac{1}{2}(p+q)} e^p = e^{\frac{1}{2}(p-q)} , \quad \text{and} \\ \alpha_1(1) &= e^{-P\Delta} d_1 = e^{-\frac{1}{2}(p+q)} e^q = e^{\frac{1}{2}(q-p)} . \end{aligned}$$

From (3) we conclude that

$$\begin{aligned} &\left\| e^{-nP\Delta} \mathbb{L}_{\Delta}^n(\psi) - \left(\alpha_0(n \bmod m) h_{n \bmod m} \psi(1, 2, 1, 2, \dots) \right. \right. \\ &\quad \left. \left. + \alpha_1(n \bmod m) h_{n+1 \bmod m} \psi(2, 1, 2, 1, \dots) \right) \right\|_{\Sigma_A^+} \xrightarrow{n \rightarrow \infty} 0 , \end{aligned}$$

for all $\psi \in \mathcal{C}(\Sigma_A^+)$. An interesting fact is that, putting $f = g = \mathbf{1}$ in (4) and putting $\psi = \mathbf{1}$ in the above expression we have

$$\begin{aligned} \mu(\Delta_n) &= \int_{\Delta_n} d\mu = \int \mathbb{L}_\Delta^n(\mathbf{1}) d\mu \\ &= e^{nP_\Delta} \left(\alpha_0(n \bmod m) \int h_{n \bmod m} d\mu \right. \\ &\quad \left. + \alpha_1(n \bmod m) \int h_{n+1 \bmod m} d\mu \right) + o(e^{nP_\Delta}). \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{n \rightarrow \infty} e^{-2nP_\Delta} \mu(\Delta_{2n}) &= \alpha_0(0) \int h_0 d\mu + \alpha_1(0) \int h_1 d\mu \\ &= \int (h_0 + h_1) d\mu = \int h_\Delta d\mu, \end{aligned}$$

but

$$\begin{aligned} \lim_{n \rightarrow \infty} e^{-(2n+1)P_\Delta} \mu(\Delta_{2n+1}) &= \alpha_0(1) \int h_1 d\mu + \alpha_1(1) \int h_0 d\mu \\ &= e^{\frac{1}{2}(q-p)} \int h_0 d\mu + e^{\frac{1}{2}(p-q)} \int h_1 d\mu. \end{aligned}$$

The latter is not in general equal to $\int h_\Delta d\mu$ if $p \neq q$ (see explicit example below). Therefore $\lim_{n \rightarrow \infty} e^{-nP_\Delta} \mu(\Delta_n)$ may not exist in general. However, if $p = q$ then $\alpha_j(k) = 1$ for all j, k and then the limit is given by

$$\lim_{n \rightarrow \infty} e^{-nP_\Delta} \mu(\Delta_n) = \int h_\Delta d\mu.$$

Note also that even when $p \neq q$ there are choices of normalised potential φ such that

$$\lim_{n \rightarrow \infty} e^{-nP_\Delta} \mu(\Delta_n) = e^{\frac{1}{2}(q-p)} \int h_0 d\mu + e^{\frac{1}{2}(p-q)} \int h_1 d\mu = \int h_\Delta d\mu.$$

For explicit examples of the above remarks, take for instance φ defined by $\varphi|_{C[1]_0} \equiv p$ and $\varphi|_{C[2]_0} \equiv q$. Then necessarily h_0 is equal to 1 on the cylinders $C[1]_0$ and $C[3]_0$, and it is equal to 0 on $C[2]_0$. Similarly, h_1 is equal to 1 on $C[2]_0$ and $C[3]_0$, and it is equal to 0 on $C[1]_0$. Therefore

$$\begin{aligned} \int h_\Delta d\mu &= (\mu(C[1]_0) + \mu(C[3]_0)) + (\mu(C[2]_0) + \mu(C[3]_0)) \\ &= (1 - \mu(C[2]_0)) + (1 - \mu(C[1]_0)). \end{aligned}$$

Now the condition of φ being normalised implies that the values of φ on the cylinder $C[3]_0$ is uniquely determined. In fact on this cylinder φ is the 2-step cylindrical function given by

$$\varphi|_{C[31]_0} \equiv \log(1-e^q), \quad \varphi|_{C[32]_0} \equiv \log(1-e^p), \quad \text{and} \quad \varphi|_{C[33]_0} \equiv \log(1-e^p-e^q).$$

Hence μ is the Markov measure defined by the stochastic matrix

$$P = \begin{pmatrix} 0 & e^q & 1 - e^q \\ e^p & 0 & 1 - e^p \\ e^p & e^q & 1 - e^p - e^q \end{pmatrix}.$$

This matrix has the stationary strictly positive left eigenvector (p_1, p_2, p_3) given by

$$(p_1, p_2, p_3) = \left(\frac{e^p}{1 + e^p}, \frac{e^q}{1 + e^q}, 1 - \frac{e^p}{1 + e^p} - \frac{e^q}{1 + e^q} \right).$$

Therefore, $\mu(C[i]_0) = p_i$ for $i = 1, 2, 3$, which implies that

$$\int h_\Delta d\mu = \left(1 - \frac{e^p}{1 + e^p} \right) + \left(1 - \frac{e^q}{1 + e^q} \right) = \frac{2 + e^p + e^q}{(1 + e^p)(1 + e^q)}.$$

Now we compare the above expression with

$$\begin{aligned} e^{\frac{1}{2}(q-p)} \int h_0 d\mu + e^{\frac{1}{2}(p-q)} \int h_1 d\mu &= e^{\frac{1}{2}(q-p)} (1 - p_2) + e^{\frac{1}{2}(p-q)} (1 - p_1) \\ &= \frac{e^{\frac{1}{2}(q-p)}}{1 + e^q} + \frac{e^{\frac{1}{2}(p-q)}}{1 + e^p} = \frac{e^{\frac{1}{2}(q-p)} (1 + e^p) + e^{\frac{1}{2}(p-q)} (1 + e^q)}{(1 + e^p)(1 + e^q)}. \end{aligned}$$

The two expressions coincide if and only if

$$2 + e^p + e^q = e^{\frac{1}{2}(q-p)} (1 + e^p) + e^{\frac{1}{2}(p-q)} (1 + e^q).$$

Introducing $a = e^{\frac{1}{2}(q-p)}$ we see that

$$2 + e^p + a^2 e^p = a (1 + e^p) + a^{-1} (1 + a^2 e^p),$$

which is equivalent to

$$a (a^2 - 2a + 1) e^p = a^2 - 2a + 1.$$

Since $a e^p = e^{\frac{1}{2}(p+q)} = e^{P\Delta} < 1$, the above equality holds if and only if $a = 1$ (i.e. if $p = q$). Therefore, whenever $p \neq q$, we have

$$\int h_\Delta d\mu \neq e^{\frac{1}{2}(q-p)} \int h_0 d\mu + e^{\frac{1}{2}(p-q)} \int h_1 d\mu.$$

Hence, $e^{-nP\Delta} \mu(\Delta_n)$ does not converge as $n \rightarrow \infty$ when $p \neq q$.

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